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FINITE ELEMENT ANALYSIS OF FREE SURFACE FLOW THROUGH GATES

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SUMMARY

Two-dimensional finite element analyses of two types of gate—(i) a conduit gate with pressure flow upstream of the gate and free surface flow downstream of the gate and (ii) a sluice gate with free surfaces both upstream and downstream of the gate—are done using ideal fluid theory. The conduit gate problem is solved using both ϕ- and ψ-formulations. Various methods of satisfying the boundary conditions were tested for both formulations. The ψ-formulation developed in the present study is found to converge faster for flows with Froude numbers less than 4, which are common in sluice gates. The results obtained from the present study are compared with results from analytical and experimental techniques available in the literature. The ψ-formulation developed in the present study is then used to solve the spillway gate problem, for which no analytical solution is available.

KEY WORDS: gates; finite elements; potential flow; free surface

1. INTRODUCTION

Conduit gates, sluice gates and spillway gates are used to regulate flows in hydraulic structures. Conduit gates are usually provided at the outlet works to regulate the flow of water. The location of a conduit gate at the downstream end of a hydraulic structure causes gravity flow downstream of the gate, while a pressure flow exists upstream, thus causing a transition from conduit to free surface. The sluice gate problem differs from the conduit gate problem in two ways. First, in the sluice gate problem the free surface flow exists upstream also; secondly, the discharge is not known *a priori*. The boundary condition at the lip of the gate provides a mechanism for updating the trial discharge. Spillways are usually designed for ungated flow. Spillways with tainter gates are useful in controlling the discharge for a given design head. The hydraulic design of gated spillways is usually based on the United States Army Corps of Engineers[1] design only. The spillway gate has four degrees of freedom, namely the position of the gate trunnion, the gate radius, the gate opening and the position of the gate seat, which makes it difficult to obtain an analytical solution. The computational difficulties encountered in the discharge calculation and the satisfaction of boundary conditions in the case of a spillway gate are also discussed in this study. The three problems discussed in the present study, namely the conduit gate, the sluice gate and the spillway gate, belong to the class of problems called free boundary problems. Crank[2] gives an excellent review of free boundary problems available in the literature. In free boundary problems one of the boundaries is not known *a priori*. The two boundary conditions that the pressure is constant and that there can be no flow across the boundary have to be satisfied on the unknown free surface. Since the free surface is not physically constrained, the solution region is deformable and the shape and size of this region have to be determined as part of the solution.

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The flow under the gates is a highly converging one in which the inertia and gravity forces predominate over the viscous forces and the flowing fluid is assumed to be incompressible, homogeneous and non-viscous; hence the flow is irrotational. Thus the problem can be solved within the framework of potential flow theory. The flow is therefore governed by the Laplace equation with appropriate boundary conditions.

2. REVIEW OF LITERATURE

Rouve and Abdul Khader[3] solved the problem of the two-dimensional transition from conduit to free surface using complex variable theory and the conformal mapping technique. Laboratory experiments were also conducted by them to compare the results from the analytical solution. Fangmeier and Strelkoff[4] solved the sluice gate problem by taking into account the effects of the upstream free surface and gravity exactly. However, their solution is restricted to sluice gates with rectilinear boundaries only. Larock[5] developed a theory for planar sluice gates of arbitrary inclination. The effect of the upstream free surface was neglected by assuming a horizontal fixed boundary and the effect of gravity was not properly accounted for. Larock[6] extended his work on planar sluice gates to radial gates and gives what is perhaps the only analytical solution available for radial gates. Southwell and Vaisey[7] solved the sluice gate problem using finite differences. Masliyah *et al.*[8] used a boundary-fitted co-ordinate system to fit the shape of the free surface with one of the co-ordinate lines. Finite element solutions of the sluice gate problem began with McCorquodale and Li.[9] The method is similar to the one used by Southwell and Vaisey[7] but uses finite elements. However, the large number of trials required in determining the free surface is one of the disadvantages of this approach. The papers by Chan and Larock[10] and Chan *et al.*[11] are considered to be an important contribution to the finite element solution of potential flows with a free surface. Chan and Larock[10] extended the work of Chan *et al.*[11] to solve axisymmetric problems such as orifices and valves. Isaacs[12] solved the sluice gate problem by the ψ-formulation using curved cubic triangular elements, taking into account the effect of the upstream free surface. Diersch *et al.*[13] extended the work of Chan *et al.*[11] to solve the sluice gate and spillway problems by the ϕ-formulation using quadratic triangular elements. The effect of the upstream free surface is not taken into account and hence the method is valid only for a limited range of gate opening/total head ratios. Heng *et al.*[14] solved the sluice gate and radial gate problems using the ψ-formulation with quadratic triangular elements. Suresh Rao and Sankaranarayanan,[15] closely following the Chan *et al.*[11] formulation, solved the flow through a conduit gate first using eight-noded quadrilateral elements composed of four triangular elements and then using an equal number of eight-noded isoparametric elements and concluded that the former predict velocities and pressures more accurately. A second set of investigators[16–19] used a novel variable domain functional in combination with finite elements and solved either a crump weir or a spillway problem using the streamfunction formulation. Larock[20] analysed the flow over gated spillway gates, for a known discharge, by using two different methods, namely complex function theory and the finite element method. Finnie and Jeppson[21] used a commercial finite element computer code to solve the sluice gate problem by using turbulent flow theory and compared the velocities and pressures obtained with the measured values obtained by them from a model sluice gate. Cheng *et al.*[22] solved the sluice gate problem using the boundary integral equation method (BIEM).

3. MATHEMATICAL FORMULATIONS FOR CONDUIT GATE PROBLEM

Figure 1(a) shows the definition sketch of a two-dimensional conduit gate with a top hood angle of 90°. A uniform flow is assumed to approach from AB with a thickness a and converge towards the opening of depth b. From there onwards a free surface emerges and contracts along DE to a final depth y,

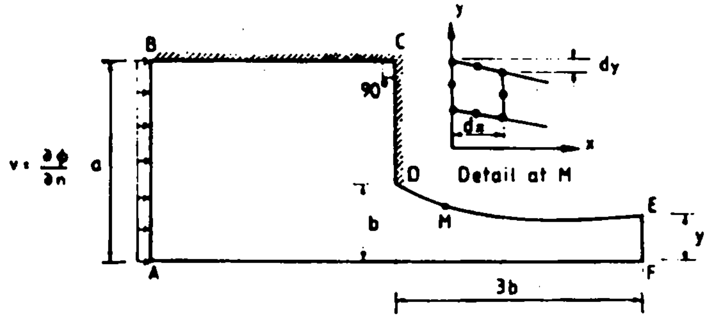

Figure 1(a). Definition sketch

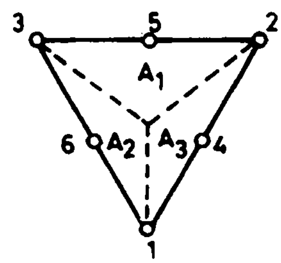

Figure 1(b). Typical top triangular element

resulting in a coefficient of contraction $C_c = y/b$. The asymptotic free surface elevation is assumed to occur at a distance equal to three times the slot opening b. The conduit is assumed to be wide enough to approximate the flow as two-dimensional. The problem is formulated in terms of an unknown head for a given discharge and gate opening.

3.1. *Differences between ϕ- and ψ-formulations*

Table I gives the mathematical formulations of the problem using ϕ and ψ. In the case of the ϕ-formulation the constant normal velocity equation (3) is imposed upstream, while a constant potential is applied along the upstream and downstream faces to satisfy the far-upstream and far-downstream uniform velocities. In the case of the ψ-formulation, equation (9) is imposed to satisfy the uniform velocities upstream and downstream. Hence the second term in equation (8) reduces to zero when the upstream and downstream uniform velocities are imposed in the ψ-formulation. The major difference between the ϕ- and ψ-formulations lies in the application of the constant pressure condition. Assuming a trial free surface, velocities are calculated along the free surface using equation (5) and used to satisfy the constant pressure condition. The velocity potentials are calculated along the free surface assuming a linear variation in ϕ between two successive nodes starting from the node far downstream in the ϕ-formulation. In the case of the ψ-formulation the velocities are directly implemented as a Neumann boundary condition. When solving the problem using eight-noded arbitrary quadrilateral elements, the quadratic distribution of velocities is incorporated as a Neumann boundary condition. This leads to faster convergence in the case of the ψ-formulation. As stated earlier, in free boundary problems the two boundary conditions that the pressure is constant and that there can be no flow across the boundary have to be specified. The application first of zero normal velocity, equation (6) of Table I, causes the second term in equation (2) of Table I to become zero, thus resulting in the satisfaction of only one boundary condition after each iteration. This sequence of satisfying the boundary conditions is not

Table I. Mathematical modelling of ϕ- and ψ-formulations

ϕ-Formulation		ψ-Formulation	
Governing equation			
$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$	(1)	$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$	(7)
Functional			
$I(\phi)=\frac{1}{2}\int\int\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y - \int\phi\frac{\partial\phi}{\partial n}\mathrm{d}s$	(2)	$I(\psi)=\frac{1}{2}\int\int\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y - \int\psi\frac{\partial\psi}{\partial n}\mathrm{d}s$	(8)
$u=\frac{\partial\phi}{\partial n}$ along AB	(3)	$v=\frac{\partial\psi}{\partial s}=0$ along AB and EF	(9)
$\phi=\text{constant}$ along EF $p=\text{constant}$ along DE	(4)		
$\frac{V_i^2}{2}+gy_i=\frac{V_E^2}{2}+gy_E$ where $V=\frac{\partial\phi}{\partial s}$	(5)	$\frac{V_i^2}{2}+gy_i=\frac{V_E^2}{2}+gy_E$ where $V=\frac{\partial\psi}{\partial n}$	(10)
$V_n=\frac{\partial\phi}{\partial n}=0$ along DE	(6)	$V_n=\frac{\partial\psi}{\partial s}=0$ along DE	(11)
		$\psi=0$ along AF $\psi=Q$ along BCDE	(12)

Note: i is any node on the free surface and E is the node far downstream on the free surface.

successful in high-speed free surface flow problems owing to the instabilities caused by higher velocities. It was found to be more successful by Suresh Rao and Sankararayanan[23] in solving free boundary problems in porous media owing to the low velocities prevailing in seepage flow. In solving free surface problems in hydraulic structures, assuming a trial free surface, the constant pressure condition is satisfied first. Then the problem is solved for the resulting fixed domain. Next the condition of zero normal velocity is satisfied by fitting a series of cubic polynomials through each set of three successive nodal points as described by Chan *et al.*[11] This process is repeated till the convergence criterion is satisfied. Owing to the high velocities prevailing in the case of free water surface problems, quadratic elements are used such that the field variable is approximated by a second-order polynomial, so that the approximations for the velocity components are linear within each triangular region. Chan *et al.*[11] and Brebbia and Connor[24] give the element and load matrices for a quadratic triangular element. In the case of quadrilateral elements composed of four triangular elements, the coefficients corresponding to the four triangular elements are assembled. Then the matrix coefficients corresponding to the five interior nodes are eliminated using static condensation as described by Cook *et al.*[25]

3.2. Method of satisfaction of constant pressure condition

3.2.1. Method of satisfaction of constant pressure condition in ϕ-formulation. The technique adopted here to solve the conduit gate problem is essentially the same as that given by Chan *et al.*[11] By applying the Bernoulli equation between any node i and the downstream reference point d (Figure 1(a)), on the free surface

$$\frac{V_i^2}{2}+\frac{p_i}{\rho}+gy_i=\frac{V_d^2}{2}+\frac{p_d}{\rho}+gy_d, \tag{13}$$

where V is the velocity, p is the pressure, g is the acceleration due to gravity, y is the depth of water, ρ is the density of water.

The value of ϕ at any node i is[11]

$$\phi_i=\frac{V_i+V_j}{2}\sqrt{[(\Delta x)^2+(\Delta y)^2]}+\phi_j. \tag{14}$$

The values of ϕ are calculated by assuming the value of ϕ at the farthest node, ϕ_j, and then calculating the values of ϕ up to the lip using equation (14). Thus the constant pressure condition is satisfied by specifying the values of velocity potentials along the free surface.

3.2.2. Method of satisfaction of constant pressure condition in ψ-formulation. A new method of satisfaction of the constant pressure condition in the ψ-formulation is used in the present study and found to possess better convergence properties. The velocities are calculated using equation (13) and directly imposed as a Neumann boundary condition.

The second term in equation (8) of Table I is given by

$$\{R\}=\int\psi\frac{\partial\psi}{\partial n}\,\mathrm{d}s. \tag{15}$$

Knowing the tangential velocities $(\partial\psi/\partial n)$ and assuming a quadratic variation in velocities and streamfunction, the integral is derived for a typical case.

A typical top triangular element of the eight-noded quadrilateral element is shown in Figure 1(b). Taking the shape functions of a six-noded triangle for ψ and a one-dimensional quadratic distribution for velocities, equation (15) becomes

$$\{R\}=\int\left\{\begin{matrix}0\\L_2(2L_2-1)\\L_3(2L_3-1)\\0\\4L_2L_3\\0\end{matrix}\right\}[L_2(1-2L_3)V_2+(4L_2L_3)V_5+L_3(2L_3-1)V_3]\,\mathrm{d}s. \tag{16}$$

The integral corresponding to node 2 is given by

$$\{R\}=\int L_2^2(2L_2-1)(1-2L_3)V_2\,\mathrm{d}s+\int L_2(2L_2-1)(4L_2L_3)V_5\,\mathrm{d}s+\int L_2(2L_2-1)L_3(2L_3-1)V_3\,\mathrm{d}s. \tag{17}$$

Integrals of the type given by equation (17) can be evaluated using the formula

$$\int L_1^aL_2^b\,\mathrm{d}l=\frac{a!b!}{(a+b+1)!}l. \tag{18}$$

The first integral in equation (17) is derived as

$$\int L_2^2(2L_2-1)(1-2L_3)V_2\,\mathrm{d}s=\left(\frac{2\times 3!}{4!}-\frac{2!}{3!}-\frac{4\times 3!\times 1!}{5!}+\frac{2\times 2!\times 1!}{4!}\right)l_{23}V_2$$
$$=\frac{4}{30}l_{23}V_2 \tag{19}$$

by applying the Gauss–Legendre relation (18). Thus the integral in equation (15) can be evaluated as

$$\{R\}=l_{23}\left\{\begin{matrix}0\\ \frac{4}{30}V_2+\frac{1}{15}V_5-\frac{1}{30}V_3\\ -\frac{1}{30}V_2+\frac{1}{15}V_5+\frac{2}{15}V_3\\ 0\\ \frac{1}{15}V_2+\frac{8}{15}V_5+\frac{1}{15}V_3\\ 0\end{matrix}\right\}. \tag{20}$$

Chan *et al.*[11] used a constant upstream velocity as the only Neumann boundary condition when solving the orifice problem using the ϕ-formulation. However, in the present study the quadratic distribution of velocities is incorporated while satisfying the constant pressure condition in the case of the ψ-formulation. If a linear velocity distribution ($V_5=(V_2+V_3)/2$) is assumed on the sides, the integral equation (15) is evaluated as

$$\{R\}=\frac{l_{23}}{6}[0\;\; V_2\;\; V_3\;\; 0\;\; 2(V_2+V_3)\;\; 0]^{\mathrm{T}}. \tag{21}$$

For a constant velocity distribution ($V_3=V_2=V_5=V$)

$$\{R\}=\frac{l_{23}}{6}V[0\;\;1\;\;1\;\;0\;\;4\;\;0]^{\mathrm{T}}. \tag{22}$$

Thus equation (22) can be used as the load vector for the integral corresponding to constant velocities upstream in the ϕ-formulation.

3.3. *Method of satisfying zero normal velocity on free surface*

Assuming a trial free surface, velocities are calculated and implemented as a Dirichlet boundary condition in the ϕ-formulation and as a Neumann boundary condition in the ψ-formulation. The problem is solved for the resulting fixed domain. Then velocities are calculated from the ϕ- or ψ-distribution as

$$u=\partial\phi/\partial x,\qquad v=\partial\phi/\partial y,\qquad u=\partial\psi/\partial y,\qquad v=-\partial\psi/\partial x. \tag{23}$$

For satisfying zero normal velocity ($v_n=0$),

$$u\sin\theta+v\cos\theta=0, \tag{24}$$

$$S=\tan\theta=-v/u. \tag{25}$$

Consider an element on the free surface defined by

$$y=Ax^3+Bx^2+Cx+D. \tag{26}$$

The free surface correction is[11]

$$\Delta y=\frac{S_1+4S_2+S_3}{6}\Delta x, \tag{27}$$

where $S_i = v_i/u_i$ for $i = 1, 2, 3$. When the slope of the lip is nearly vertical, v_1/u_1 becomes excessively large. Thus equation (26) has to be modified as

$$x = Ay^3 + By^2 + Cy + D. \tag{28}$$

Similarly, the equation for the free surface correction is[11]

$$\Delta y = \frac{6\Delta x}{S_1 + 4S_2 + S_3}, \tag{29}$$

where $S_1 = u_1/v_1 = -\tan\alpha$, $S_2 = u_2/v_2$, $S_3 = u_3/v_3$ and α is the acute angle between the gate and the y-axis. Thus the condition of zero normal velocity is satisfied by fitting a set of cubic polynomials through each set of three successive nodal points.

The net correction Δy is distributed along each vertical, i.e. the co-ordinates of the corner nodes along a vertical are corrected by a value

$$\Delta y_i = \frac{\Delta y}{y_{CN} - y_1}(y_i - y_1), \quad i = 1, \ldots, CN, \tag{30}$$

where CN is the number of corner nodes along a vertical. The above method of correcting the free surface[11] is used in the present study to avoid coarsening or overlapping of the free surface nodes.

3.4. *Convergence criteria*

When solving the transition problem through the ϕ-formulation, either of two convergence criteria can be specified:

(i) the difference in contraction coefficient between successive iterations, i.e. $|C_c^k - C_c^{k-1}| \leqslant \epsilon$, where C_c^k is the contraction coefficient at iteration k and ϵ is the error tolerance
(ii) the difference in ordinates of the free surface nodes between successive iterations for all free surface nodes, i.e. $|y_i^k - y_i^{k-1}| \leq \epsilon$ for $i = 1, \ldots, NSN$, where y_i^k is the y-co-ordinate of node i at iteration k, NSN is the number of free surface nodes and ϵ is the error tolerance.

When solving the problem through the ψ-formulation, in addition to the above two boundary conditions, the difference between the ψ-values and the discharge for all the free surface nodes can also be specified as a convergence criterion, i.e. $|\psi_i^k - Q| \leq \epsilon$, where ψ_i^k is the ψ-value of node i at iteration k, Q is the discharge and ϵ is the error tolerance.

4. NUMERICAL SIMULATION FOR CONDUIT GATE PROBLEM

Figure 2(a) shows the discretization for a typical two-dimensional 45° transition from conduit to free surface. Eight-node arbitrary quadrilateral elements (Figure 2(b)) composed of four quadratic triangular elements are used. The discretization is made finer in the vicinity of the gate to take care of the higher velocity gradients that prevail in this region. The discretization is also made finer near the free surface for better approximation of velocities. The domain marked ABCDE in Figure 2(a) is fixed and the matrix corresponding to the elements in this domain is calculated only once. However, the flow region marked DEFG is a variable domain and the stiffness matrix corresponding to the elements in this domain is calculated every time an adjustment of the free surface is made. Theoretically, for an ideal fluid the asymptotic free surface is assumed to occur at infinity. In the present study also the asymptotic free surface is assumed to occur at a distance equal to three times the slot opening from the slot.

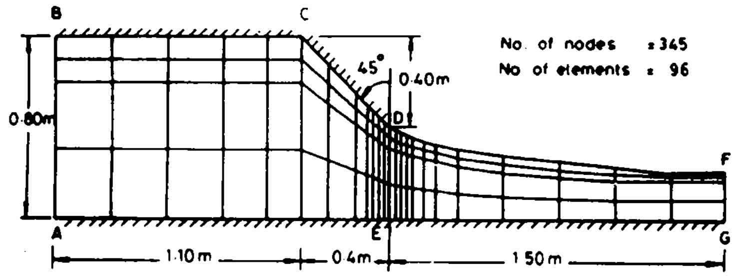

Figure 2(a). Typical discretization for conduit gate problem

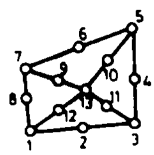

Figure 2(b). Eight-noded quadrilateral element composed of four triangular elements

The velocities are calculated only for the top triangular element formed from the eight-noded quadrilateral element along the free surface elements only, as described in Reference 26. The problem is solved using both ϕ- and ψ-formulations. Table II gives the contraction coefficients for three different types (Table III) of discretization. Eight-noded quadrilateral elements composed of four triangular elements and eight-noded isoparametric elements are used in the domain discretization. Discretization I consisted of 219 nodes and 60 elements. Discretization II consisted of 345 nodes and 96 elements. The contraction coefficients obtained in the present study are closer to the analytical solution given by Rouve and Abdul Khader[3] with finer discretization near the gate opening. The contraction coefficients obtained using eight-noded quadratic elements composed of triangular elements (Discretization II) are found to be closer to the analytical solution than those obtained using an equal number of eight-noded isoparametric elements (Discretization III). The contraction coefficients for zero gravity flow are closer to those for flows with Froude numbers greater than 8. Hence it may be seen that the effect of gravity is negligible for flows with Froude numbers greater than 8, as concluded by Rouve and Abdul Khader.[3]

The downstream free surface obtained for a Froude number of 4 shows close comparison with the analytical solution of Rouve and Abdul Khader[3] as seen in Figure 3. Table IV gives the maximum energy deficits along the free surface, i.e. the maximum deviations of the total head H along the free surface. The maximum energy deficit is defined as the ratio of the maximum deviation of the total head along the free surface to the final asymptotic total head. The energy deficits calculated are well within the acceptable limits. It is also seen that the energy deficit is very small in the case of quadrilateral

Table II. Comparison of contraction coefficients

Sl. No.	Downstream Froude number	Analytical Solution*	Dis I (ϕ)	Dis II (ϕ)	Dis II (ψ)	Dis III (ϕ)
1	2	0·7067	0·7133	0·704	0·703	—
2	4	0·7413	0·7542	0·743	0·741	0·753
3	6	0·7467	0·7600	0·748	—	—
4	8	0·7547	0·7610	0·752	—	—

* Rouve and Khader (1969).

Table III. Types of discretization

Discretization	Nodes	Elements	Element Type
I	219	60	Eight noded quadrilateral
II	345	96	Eight noded quadrilateral
III	345	96	Eight noded isoparametric element

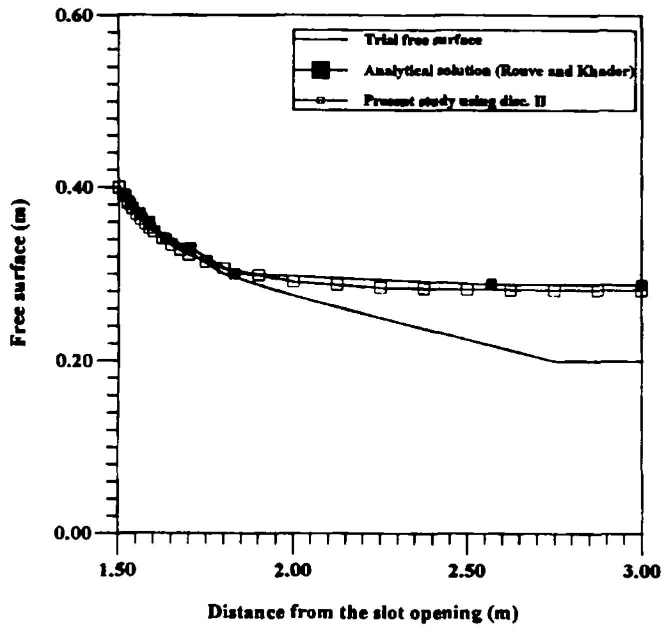

Figure 3. Comparison of free surfaces downstream of gate

Table IV. Maximum energy deficits

Froude number far downstream	Total head (m)	Slot opening / total head	Energy deficit	
			Discr.-II	Discr.-III
2	0·868	0·231	0·127%	
4	2·680	0·150	0·256%	3·5%
6	5·760	0·070	0·286%	

elements composed of triangular elements, thus indicating the accurate prediction of velocities in such an element.

5. COMPARATIVE MERITS OF USING ϕ- AND ψ-FORMULATIONS

In the conduit gate problem the velocities are directly imposed as a Neumann boundary condition when solving the problem through the ψ-formulation, but the velocities are implemented as a Dirichlet boundary condition in the ϕ-formulation in order to satisfy the constant pressure condition. The direct imposition of velocities in the case of the ψ-formulation leads to faster convergence compared with the ϕ-formulation as shown in Figure 4. However, this faster convergence is found to be limited to flows

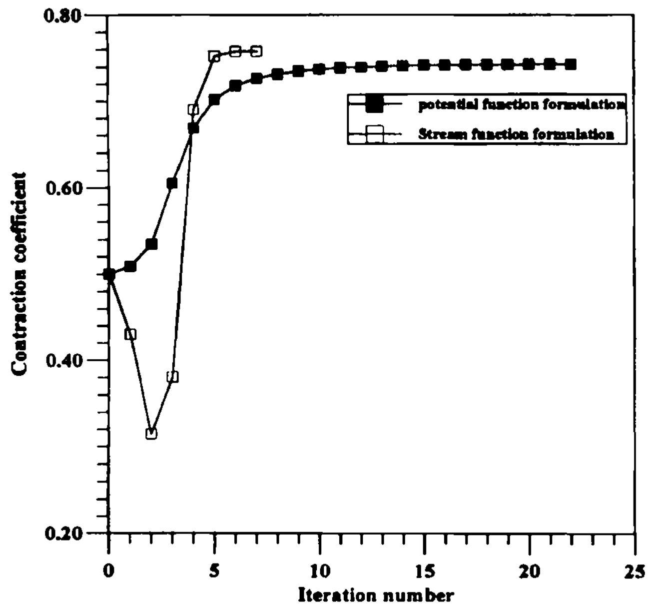

Figure 4. Convergence behaviour of ϕ- and ψ-formulations

with Froude numbers less than 4. Flows downstream with Froude numbers greater than 1 and less than 4 are generally found to occur in sluice gates and the ψ-formulation developed in the present study is adopted to solve the sluice gate problem. The ϕ-formulation is found to be applicable for a wide range of Froude numbers, but it required 25–30 iterations as compared with five to seven iterations taken by the ψ-formulation. The velocities were found to be constant along the conduit bottom at some distance from the opening and along the downstream face, validating the assumption that the free surface is asymptotic at or beyond a distance equal to three times the slot opening.

6. MATHEMATICAL MODELLING OF FLOW UNDER A SLUICE GATE

The flow under a sluice gate (Figure 5) is assumed to be two-dimensional, incompressible, inviscid and hence irrotational. The governing equation is the Laplace equation in the streamfunction $\psi(x, y)$ given by equation (7) of Table I, with the boundary conditions

$$\psi = 0 \quad \text{along AF}, \tag{31}$$

$$\psi = Q \quad \text{along BC and CD}, \tag{32}$$

$$\partial\psi/\partial n = 0 \quad \text{along AB and EF}, \tag{33}$$

$$\partial\psi/\partial n = V = \sqrt{[2g(H-y)]} \quad \text{along DE}, \tag{34}$$

$$V^2/2g + y = H \quad \text{along BC}, \tag{35}$$

$$V_n = \partial\psi/\partial s = 0 \quad \text{and} \quad \psi = Q \quad \text{along DE}, \tag{36}$$

$$v_{\mathrm{al}} = \sqrt{[2g(H-b)]} \quad \text{at D}. \tag{37}$$

Equation (7) is solved with the set of boundary conditions (31)–(37), where V is the velocity along the streamline, g is the gravitational acceleration, H is the total head, y is the local elevation above the datum and V_n is the velocity normal to the streamline. For a given gate opening b and total head H the upstream and downstream free surfaces and the discharge have to be determined. The functional

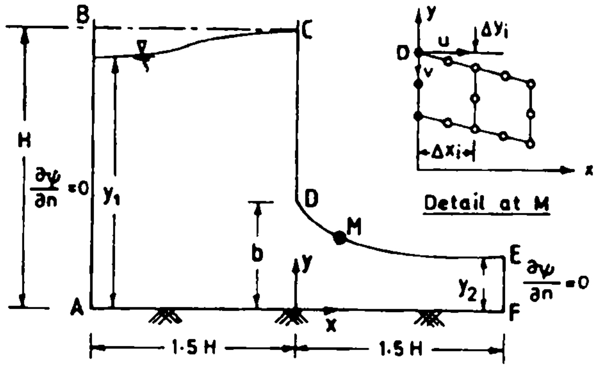

Figure 5. Flow domain for sluice gate problem

corresponding to equation (7) together with the boundary conditions is given by equation (8) of Table I.

6.1. Solution procedure

Assuming a trial discharge and trial free surface, the problem is solved for the upstream and downstream free surfaces. The initial upstream surface is assumed to be horizontal and located at a distance H from the channel floor, while a suitable trial free surface is assumed downstream. For solving the problem for free surfaces, the convergence criterion $|\psi_i^k - Q^{k_1}| \le \epsilon$ for $i=1, \ldots, NSN$ is specified, where ψ_i^k is the ψ-value for node i at iteration k, Q^{k_1} is the discharge at iteration k_1, ϵ; is the error tolerance, k is the iteration number for the free surface and k_1 is the iteration number for the discharge. After solving the problem for free surfaces for a given discharge, the velocity calculated at the lip of the gate is compared with the actual velocity calculated using equation (37). Thus the velocity at the lip of the gate, V_{al}, provides a mechanism for updating the trial discharge. If the calculated velocity does not compare well with the actual velocity, the discharge is incremented by ΔQ and the free surface is solved for Q_2 ($Q_1 + \Delta Q = Q_2$). The next trial discharge Q_3 is evaluated using the Newton–Raphson method. The recursive relation for updating the trial discharge is given by

$$Q_{i+2} = Q_i - \frac{V_{cl_i} - V_{al}}{(V_{cl_{i+1}} - V_{cl_i})/(Q_{i+1} - Q_i)}, \tag{38}$$

where V_{cl_i} is the calculated velocity at iteration i and V_{al} is the actual velocity at the lip of the gate. The iterations are continued till the calculated velocity at the lip of the gate matches the actual velocity, i.e. when $|V_{cl_i} - V_{al}| \le \epsilon$.

7. NUMERICAL SIMULATION FOR SLUICE GATE PROBLEM

In the analysis of the flow field for flow under a vertical sluice gate for a given total head, (i) the discharge, (ii) the upstream and downstream free surfaces and (iii) the pressure distribution along the floor and along the gate are usually required to be determined.

A typical discretization of the flow field for a vertical sluice gate consisted of 498 eight-noded quadrilateral elements composed of triangular elements. The results are obtained for $b/H = 0{\cdot}3$ and $0{\cdot}5$, keeping the total head H as $0{\cdot}3$ m. The contraction and discharge coefficients obtained in the

Table V. Comparison of C_c and C_d for vertical sluice gate

		Analytical solution		Present study	
b/H	Head 'H' in m	C_c	C_d	C_c	C_d
0·3	0·300	0·598	0·540	0·609	0·562
0·4	0·225	0·596	0·525	0·606	0·552
0·5	0·300	—	—	0·559	0·497

present study show close agreement with the available analytical solution as seen in Table V. Here C_c and C_d are defined as

$$C_c = Y_2/b, \qquad C_d = Q/b\sqrt{(2gy_1)}, \tag{39}$$

where y_1 and y_2 are the upstream and downstream uniform flow depths respectively, b is the gate opening and Q ($m^2\ s^{-1}$) is the exact value of the discharge satisfying the energy condition at the lip. The upstream and downstream free surfaces obtained in the present study compare well with the comprehensive analytical solution of Fangmeier and Strelkoff[4] as shown in Figures 6 and 7 respectively.

Taking the geometry of the gate the same as that given by Finnie and Jeppson,[21] the problem is solved via the ψ-formulation developed in the present study using 498 eight-noded quadrilateral elements composed of four triangular elements. Previously Suresh Rao and Sankaranarayanan[27,28] used an equal number of eight-noded isoparametric elements for the same problem. From the present study it is found that eight-noded quadrilateral elements composed of four triangular elements give a better prediction of the pressure and velocity distributions than do an equal number of eight-noded isoparametric elements, in spite of the additional computational effort involved in condensing the matrices in the former. The pressure distribution obtained in the present study compares well with that given by Finnie and Jeppson[21] as shown in Figure 8. The velocity distribution at the gate obtained in the present study compares favourably with that of Finnie and Jeppson[21] as shown in Figure 9. Finnie and Jeppson[21] analysed the sluice gate problem using turbulent flow theory, taking the initial approximation of the downstream free surface from the analytical solution by potential flow theory. The total solution time taken for the turbulent flow analysis of the sluice gate is reported to be 1 h on a Cray-2-supercomputer. However, the total CPU time required to compute all the details of the flow domain in the present study is only 300 s on the Siemens 7.580 E mainframe computer at the Indian Institute of Technology, Madras. Thus the formulations and programmes are validated for the vertical sluice gate problem.

8. NUMERICAL SIMULATION FOR SPILLWAY GATE PROBLEM

In the case of a spillway gate the geometry of the gate is defined as a function of four parameters. Hence an analytical solution satisfying the boundary conditions exactly is difficult to obtain. Since the Laplace equation has been used to model the flow, the frictional effects are neglected. However, in the experimental study also, since the scaling is done using the Froude model law, the frictional effects of the prototype are not scaled appropriately. The present study differs from that of Larock[20] by assuming the discharge to be not known *a priori*. Hence free surfaces are determined for a trial discharge and the correct discharge is determined using the boundary condition at the lip of the gate.

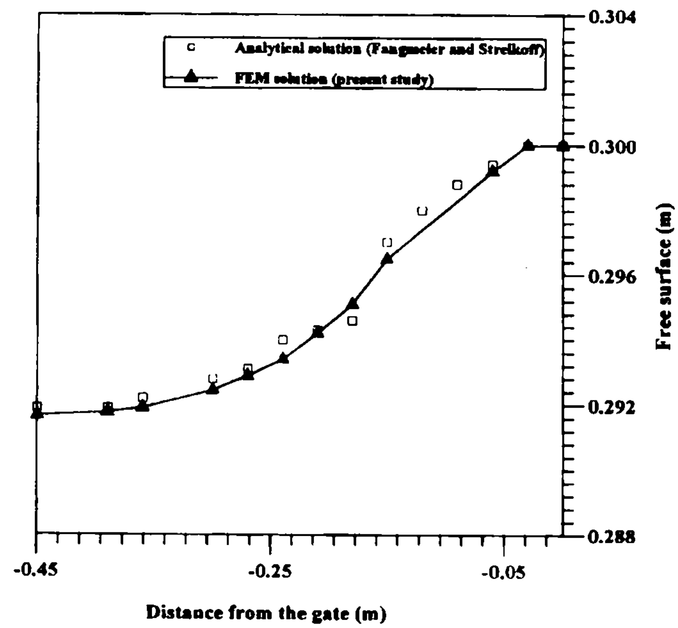

Figure 6. Comparison of upstream free surfaces

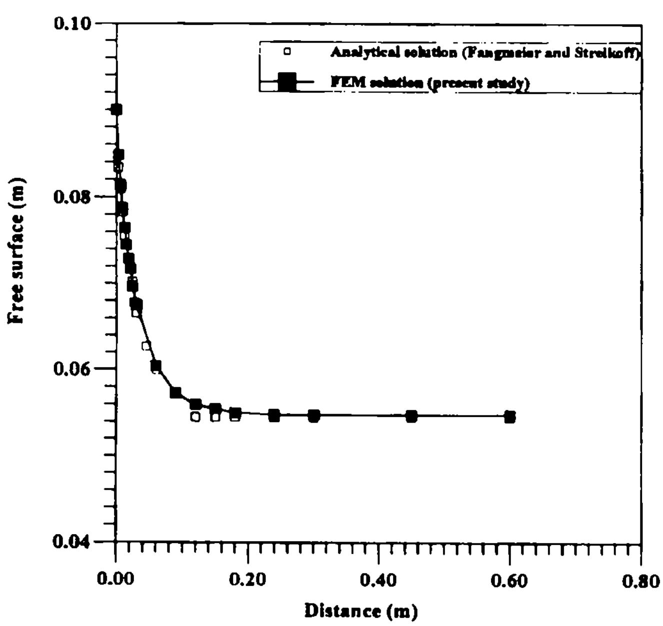

Figure 7. Comparison of downstream free surfaces

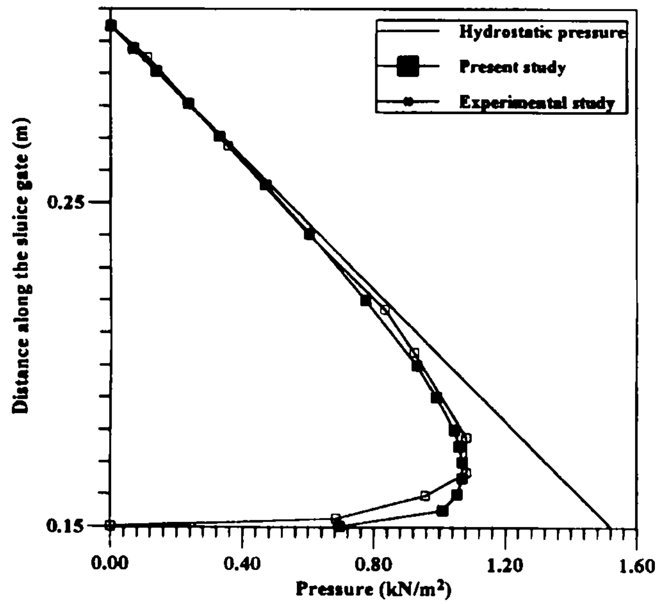

Figure 8. Comparison of pressures along sluice gate

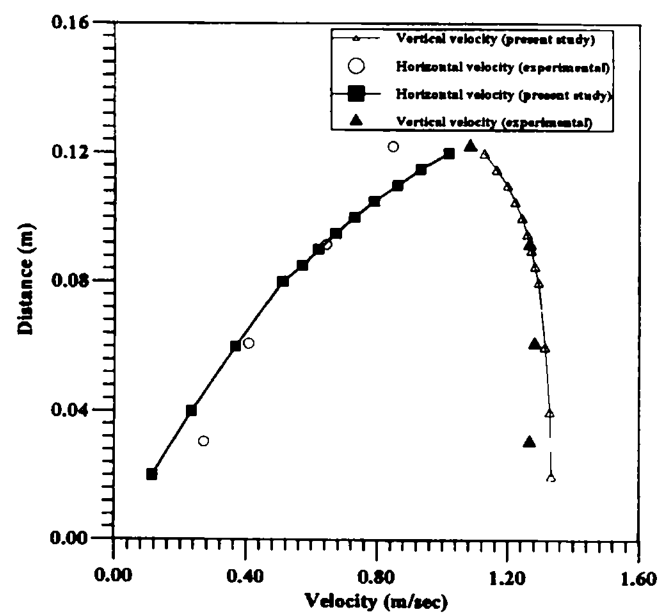

Figure 9. Comparison of velocities at gate opening

8.1. Geometrical details of the spillway gate

Figure 10 shows the definition sketch of the spillway radial gate of John Doe Dam (U.S.A.). The geometric details of the gate are as follows:[1]

design head	$H_d = 11{\cdot}3$ m
radius of gate	$R_G = 9{\cdot}38$ m
trunnion co-ordinates	$X_T = 10{\cdot}25$ m
	$Y_T = 3{\cdot}66$ m
gate lip co-ordinates	$X_L = 0{\cdot}88$ m
	$Y_L = 4{\cdot}52$ m
effective gate opening	$G_o = 4{\cdot}56$ m.

The spillway lower nappe profile is given by

$$x^{1{\cdot}85} = -2H_d^{0{\cdot}85}y. \tag{40}$$

8.2. Numerical simulation

Based on the above details, the gate angle β formed by the tangent to the gate lip and the tangent to the crest curve at the nearest point of the crest curve is taken as 91·2°. Taking the geometry of the gate

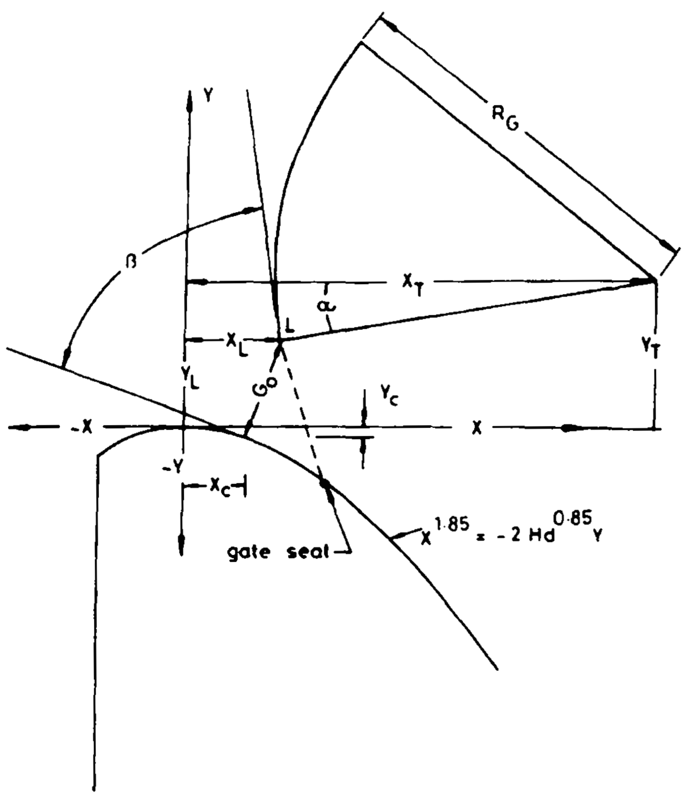

Figure 10. Definition sketch of spillway radial gate

as given in Section 8.1, the problem is solved for two different heads $H = 8{\cdot}256$ and 9·8 m, H being measured from the free surface to the highest point of the crest curve. These heads are chosen because experimental and field measurements are available for them and for a possible comparison of discharges. A typical discretization for a spillway gate problem is shown in Figure 11 consisting of 1349 nodes and 412 eight-noded quadrilateral elements. Previously Sankaranarayanan[26] used an equal number of eight-noded isoparametric elements in his study. Previous studies of the uncontrolled spillway problem show that the potential flow models represents well the actual flow in the region where the flow is rapidly contracting or acclerating. However, as the fluid passes further down the spillway face, real fluid effects become progressively more important. For this reason the downstream extent of the flow domain is delimited rather severely as shown in Figure 11.

The determination of the downstream free surface and the imposition of the uniform flow condition are made difficult by the presence of curved solid boundaries, especially the irregular bottom boundary. Uniform flow is assumed to occur at distances approximately equal to two times and one and a half times the head causing flow under the gate from the spillway face on the upstream and downstream sides respectively. The far-downstream efflux face is made normal to the spillway surface to impose the downstream uniform flow condition without much difficulty. On the other hand, other elements downstream are aligned vertically so that adjustments in the free surface are made easily. The discharges computed from the present study compare well with those computed in the prototype investigations by the United States Waterways Experimental Station as seen in Table VI. Figure 12 shows the free surface profiles downstream of the spillway for two different heads. Further details on the results for the spillway gate problem are given in Reference 26.

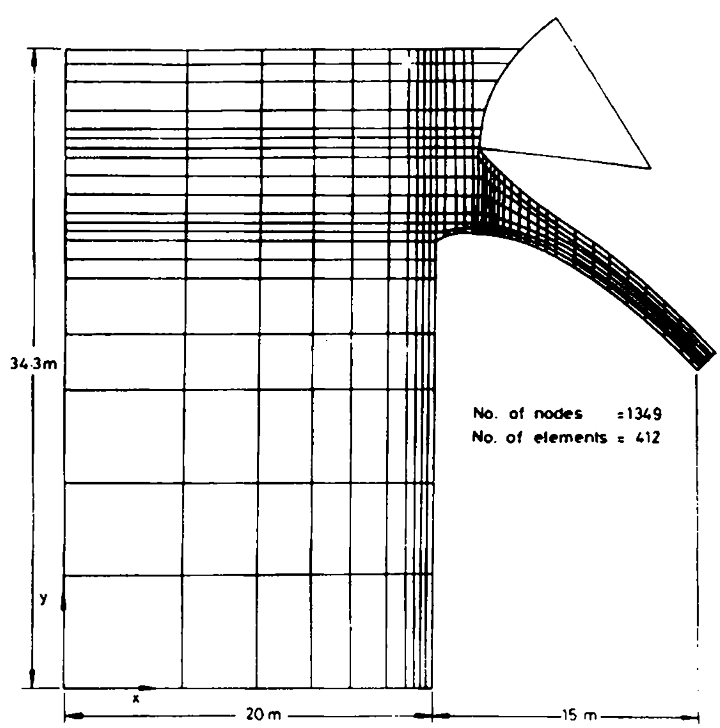

Figure 11. Typical discretization of spillway radial gate

Table VI. Comparison of discharges obtained in this study with experimental results

Head (m)	Effective gate opening (m)	Discharge per unit width (m^2/s)	
		Experiment (US Army corps)	Present study
8·256	4·56	35	34·3
9·8	4·56	39	38·85

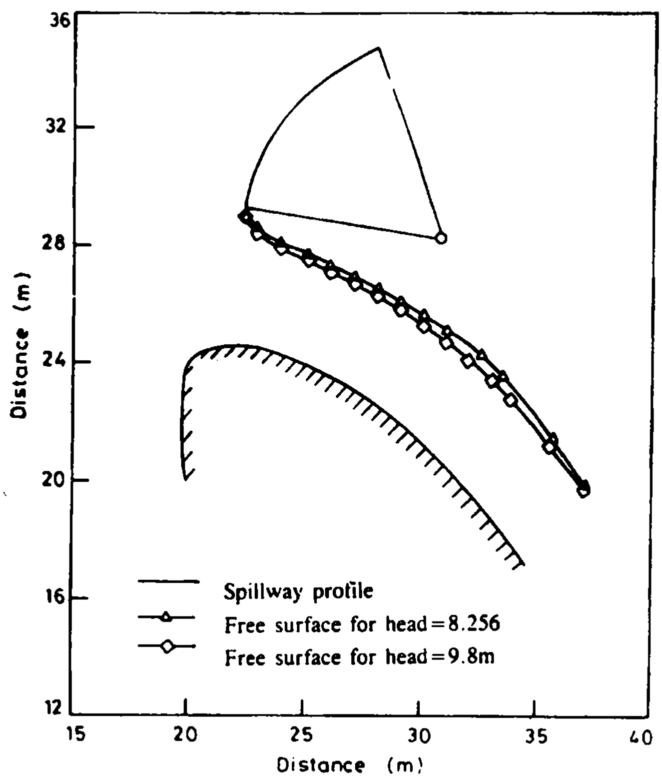

Figure 12. Free surface profiles downstream of gate for two different heads

9. CONCLUSIONS

A simple conduit gate problem with a free surface is solved by the FEM using both ϕ- and ψ-formulations. When solving the gate problem through the ψ-formulation, various combinations of satisfying the free boundary conditions are successfully tried and a new method has been evolved and is found to converge faster. The contraction coefficient and the downstream free surface obtained in the present study compare closely with the available analytical solution. The selection of the ϕ- or ψ-formulation is found to be problem-dependent. For problems in which the Froude numbers encountered are less than 4, the present study shows that the ψ-formulation converges faster than the ϕ-formulation. Incidentally, flows with Froude numbers less than 4 often occur in sluice gates. Hence the present ψ-formulation is useful for solving the sluice gate problem. A typical problem of flow under a sluice gate with a free surface upstream and downstream is solved by the FEM through the ψ-formulation and validated with available experimental and analytical solutions. With a very moderate computational time, pressure and velocity distributions are obtained which compare very well with those measured by Finnie and Jeppson,[21] thus confirming the adequacy of the algorithm used. After validating the FEM model developed for the vertical sluice gate problem, a more general problem of flow under a spillway gate with curved solid boundaries is solved.

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